Brownian disks and excursions of tree-indexed Brownian motion

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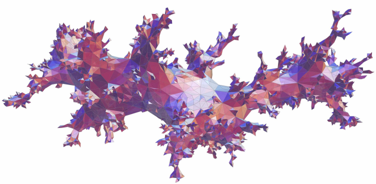
Workshop on Statistical Mechanics, Les Diablerets



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Outline

1. Brownian spheres and Brownian disks (as scaling limits of discrete planar graphs)

2. The construction of the Brownian sphere (from Brownian motion indexed by the Brownian tree)

3. Excursions of Brownian motion indexed by the Brownian tree (an analog of the classical Itô theory for Markov processes)

4. The construction of Brownian disks (from the excursion measure for Brownian motion indexed by the Brownian tree)

5. Cutting Brownian disks at heights (a remarkable growth-fragmentation process)

1. Brownian spheres and Brownian disks

Definition

A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms). Self-loops and multiple edges are allowed.

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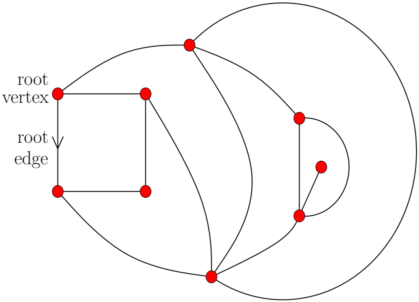

A rooted quadrangulation with 7 faces

Faces = connected components of the complement of edges

p-angulation:

- each face is incident to p edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

The Brownian sphere (or Brownian map)

Let M_n be uniform over $\mathbb{M}_n^4 = \{$rooted quadrangulations with n faces$\}$.

$V(M_n)$ vertex set of M_n

d_{gr} graph distance on $V(M_n)$

The Brownian sphere (or Brownian map)

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$d_{\rm gr}$ graph distance on $V(M_n)$

Theorem (LG 2013, Miermont 2013)

We have

$$(V(M_n), (9/8)^{1/4}\, n^{-1/4}\, d_{\rm gr}) \xrightarrow[n\to\infty]{\rm (d)} (\mathbf{m}_\infty, D)$$

in the Gromov-Hausdorff sense. The limit $(\mathbf{m}_\infty, D)$ is a random compact metric space called the Brownian sphere (or Brownian map).

Remark A similar result holds for random triangulations and for much more general random planar maps, with the same limit (Brownian sphere). For simplicity, we focus on quadrangulations in the present lecture.

Two properties of the Brownian sphere

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D) = 4 \qquad \textit{a.s.}$$

(Already "known" in the physics literature.)

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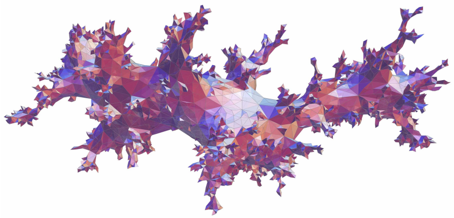
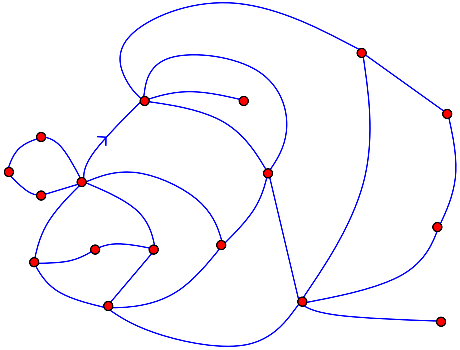
Theorem (topological type, LG-Paulin 2007)

Almost surely, $(\mathbf{m}_\infty, D)$ *is homeomorphic to the* 2*-sphere* $\mathbb{S}^2$.



Simulation: N. Curien

Quadrangulations with a boundary



A quadrangulation with a boundary of size 14.

A quadrangulation with a boundary is a rooted planar map M such that

- The root face (to the left ot the root edge) has an arbitrary even degree.
- All other faces have degree 4.

The root face is also called the outer face, and its degree is the boundary size of M.

Boltzmann quadrangulations with a boundary

For $p \geq 1$, let $\mathbb{M}^{4,p}$ be the set of all (rooted) quadrangulations with a boundary of size $2p$.
If $Q \in \mathbb{M}^{4,p}$, let $|Q|$ stand for the number of faces of Q

A Boltzmann quadrangulation with boundary size $2p$ is a random quadrangulation with a boundary $\mathbf{Q}_p$ such that :

$$\mathbb{P}(\mathbf{Q}_p = Q) = c_p \, 12^{-n} \text{ for every } Q \in \mathbb{M}^{4,p} \text{ with } |Q| = n$$

here $c_p > 0$ is the appropriate normalizing constant (depending on p).

This makes sense because

$$\#\{Q \in \mathbb{M}^{4,p} : |Q| = n\} \underset{n\to\infty}{\approx} c \, n^{-5/2} \, 12^n$$

Convergence to the Brownian disk

Recall that $\mathbf{Q}_p$ is a Boltzmann quadrangulation with boundary size $2p$. Equip the vertex set $V(\mathbf{Q}_p)$ with the graph distance $d_{\rm gr}$.

Theorem (Bettinelli and Miermont)

Then

$$\big(V(\mathbf{Q}_p),(2p/3)^{-1/2}d_{\rm gr}\big)\ \xrightarrow[p\to\infty]{\rm (d)}\ (\mathbb{D},\Delta)$$

in the Gromov-Hausdorff sense. The limit $(\mathbb{D},\Delta)$ is a random compact metric space called the free Brownian disk with perimeter 1.

By scaling one can define the free Brownian disk with perimeter r. The free Brownian disk comes with a volume measure Vol. By conditioning on $\mathrm{Vol}(\mathbb{D})=v$, one defines the Brownian disk with perimeter r and volume v.

(See also Gwynne and Miller for the simple boundary case, and Miller and Sheffield for more about Brownian disks)

Properties of the Brownian disk

Fact (Bettinelli): The free Brownian disk $\mathbb{D}$ (with perimeter $r > 0$) is homeomorphic to the closed unit disk.

Hence one can make sense of the boundary $\partial\mathbb{D}$.

The uniform measure μ on $\partial\mathbb{D}$ may be defined by the approximation

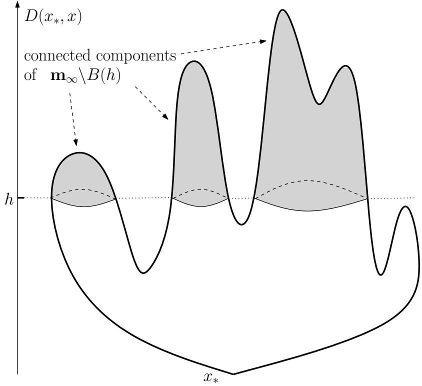
$$\langle \mu, \varphi \rangle = \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, \varphi(x)\, \mathbf{1}_{\{\Delta(x,\partial\mathbb{D})<\varepsilon\}}$$

where φ is a continuous function on $\mathbb{D}$, and $\mathrm{Vol}(\cdot)$ stands for the volume measure on $\mathbb{D}$.

In particular the total mass of μ is the perimeter (boundary size) r.

Many special subsets of the Brownian sphere $(\mathbf{m}_\infty, D)$ can be identified as Brownian disks.

Brownian disks in the Brownian sphere

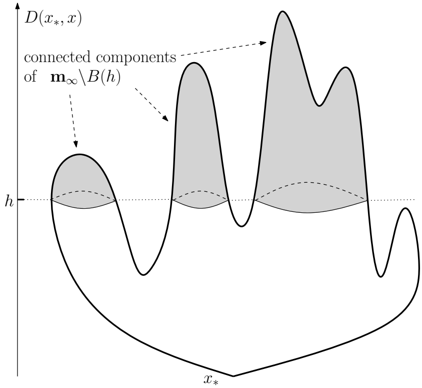


For $h > 0$, let $B(h)$ be the ball of radius h centered at the distinguished point x_* in the Brownian sphere $(\mathbf{m}_\infty, D)$

Let $\mathcal{D}_j$, $j \in J$ be the connected components of $\mathbf{m}_\infty \backslash B(h)$. We can equip each $\mathcal{D}_j$ with its intrinsic metric $D^{(j)}$

Vol : volume measure on $\mathbf{m}_\infty$

Brownian disks in the Brownian sphere



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Theorem

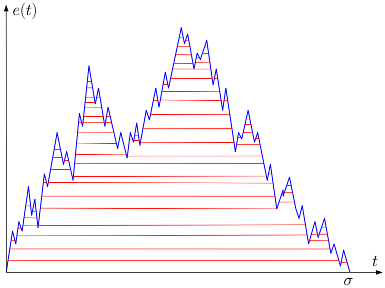
For every j, the limit

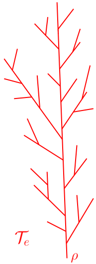
$$|\partial \mathcal{D}_j| := \lim_{\varepsilon \to 0} \varepsilon^{-2} \mathrm{Vol}\{x \in \mathcal{D}_j : D(x, \partial \mathcal{D}_j) < \varepsilon\}$$

exists, and, conditionally on $(|\partial \mathcal{D}_j|, \mathrm{Vol}(\mathcal{D}_j))_{j \in J}$*, the metric spaces* $(\bar{\mathcal{D}}_j, D^{(j)})$ *are independent Brownian disks with the prescribed volumes and perimeters.*

2. The construction of the Brownian sphere

A key ingredient: The Brownian tree, or tree coded by a Brownian excursion under $\mathbf{n}_+(\mathrm{d}e)$ (the positive Itô excursion measure).





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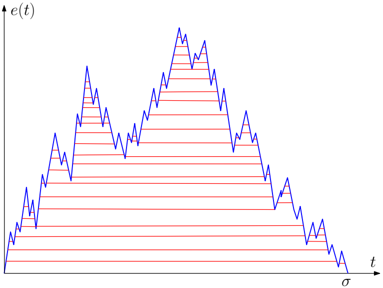

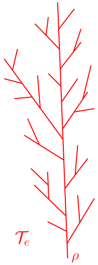

Informally, glue $s, t \in [0, \sigma]$ if they correspond to the ends of a chord drawn below the graph of e.

Formally, say that $s \sim t$ iff $e(s) = e(t) = \min_{u \in [s \wedge t, s \vee t]} e(u)$.
The Brownian tree is $\mathcal{T}_e := [0, \sigma]/\sim$, with the metric induced by

$$d_e(s, t) = e(s) + e(t) - 2 \min_{u \in [s \wedge t, s \vee t]} e(u).$$

The Brownian tree

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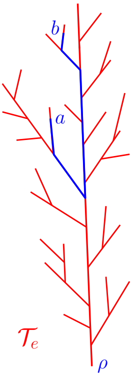
Then $(\mathcal{T}_e, d_e)$ is a compact $\mathbb{R}$-tree

(means that two points of $\mathcal{T}_e$ are connected by a unique arc $[[a,b]]$, which is isometric to a line segment — $d(a,b)$ is the length of the blue path connecting a to b)



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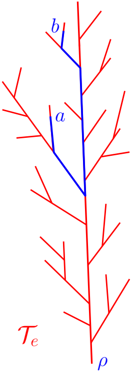

Let $p_e : [0,\sigma] \to \mathcal{T}_e = [0,\sigma]/\sim$ be the canonical projection:

- $\mathcal{T}_e$ is rooted at $\rho := p_e(0) = p_e(\sigma)$
- the volume measure Vol is the push forward of Lebesgue measure under p_e.
- the Brownian tree $\mathcal{T}_e$ also inherits a cyclic ordering from the projection p_e (it makes sense to explore the tree "clockwise" from one point to another)

Brownian motion indexed by the Brownian tree

Conditionally on $\mathcal{T}_e$, $Z = (Z_a)_{a \in \mathcal{T}_e}$ is the centered Gaussian process characterized by:

- $Z_\rho = 0$
- $\mathbb{E}[(Z_a - Z_b)^2] = d_e(a, b)$ for every $a, b \in \mathcal{T}_e$

(Technical difficulty: Z is a random process indexed by a random set. Since $\mathcal{T}_e = [0, \sigma]/\sim$, one can as well define Z as indexed by $[0, \sigma]$ — this is the Brownian snake construction)

Fact: Z has continuous sample paths.

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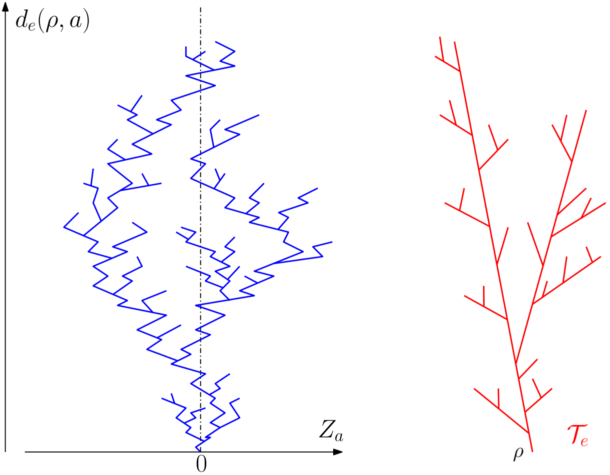
Fact: Z has continuous sample paths.

One views Z_a as a Brownian label assigned to $a \in \mathcal{T}_e$. When moving along a line segment of $\mathcal{T}_e$, labels evolve like linear Brownian motion.

Motivations for studying $\mathcal{T}_e$ and $(Z_a)_{a \in \mathcal{T}_e}$: These objects arise in a number of asymptotics for discrete models, in combinatorics, interacting particle systems, statistical physics, etc.

Brownian motion indexed by the Brownian tree 2



The collection $(Z_a)_{a \in \mathcal{T}_e}$ forms a "tree of Brownian paths" whose genealogy is prescribed by $\mathcal{T}_e$.
Z_a is also interpreted as a "label" assigned to vertex $a \in \mathcal{T}_e$.

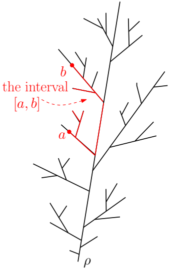
The construction of the Brownian sphere

$\mathcal{T}_e$ is the Brownian tree, $(Z_a)_{a\in\mathcal{T}_e}$ Brownian motion indexed by $\mathcal{T}_e$ (Two levels of randomness!).

Set, for every $a, b \in \mathcal{T}_e$,

$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a, b]$ is the "interval" from a to b corresponding to the cyclic ordering on $\mathcal{T}_e$ (vertices visited when going from a to b in clockwise order around the tree).



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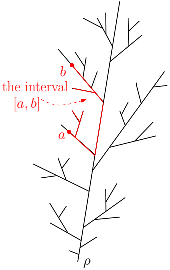

Then let D be the maximal symmetric function on $\mathcal{T}_e \times \mathcal{T}_e$ that is bounded above by D^0 and satisfies the triangle inequality. Also set

$$a \approx b \text{ if and only if } D(a,b) = 0 \quad (\text{equivalent to } D^0(a,b) = 0).$$

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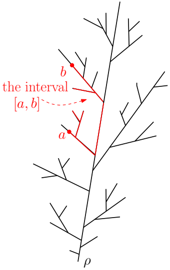

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Definition

The free Brownian sphere $\mathbf{m}_\infty$ is the quotient space $\mathbf{m}_\infty := \mathcal{T}_e/\approx$, which is equipped with the distance induced by D.

To get the "standard" Brownian sphere, condition on $\sigma = 1$.

Summary and interpretation

Starting from the Brownian tree $\mathcal{T}_e$, with Brownian labels $Z_a, a \in \mathcal{T}_e$,
$\rightarrow$ Identify two vertices $a, b \in \mathcal{T}_e$ if $D^\circ(a, b) = 0$, meaning that:

- they have the same label $Z_a = Z_b$,
- one can go from a to b around the tree (in clockwise or in counterclockwise order) visiting only vertices with label greater than or equal to $Z_a = Z_b$.

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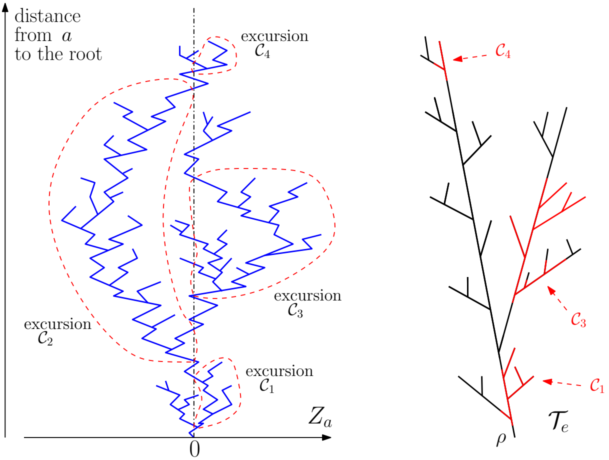
Key fact: If x_* is the vertex with minimal label ($Z_{x_*} = \min\{Z_a : a \in \mathcal{T}_e\}$) then, for every a

$$D(x_*, a) = Z_a - Z_{x_*}$$

(labels correspond to distances from x_*, up to a shift)

$\rightarrow$ conn.comp. of complement of a ball = excursions of Z above a level

3. Excursions of Brownian motion indexed by the Brownian tree



Recall:

$\mathcal{T}_e$ Brownian tree

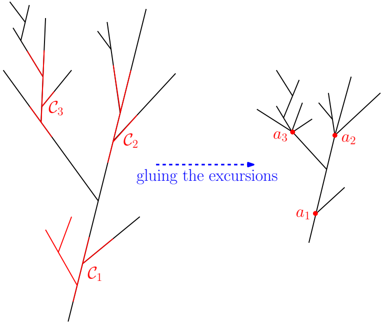
$(Z_a)_{a\in\mathcal{T}_e}$ Brownian motion indexed by $\mathcal{T}_e$

Let $(\mathcal{C}_i)_{i\in I}$ be the connected components of $\{a \in \mathcal{T}_e : Z_a \neq 0\}$.

The excursions of Z are $\big(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i}\big)$, $i \in I$, viewed as $\mathbb{R}$-trees equipped with continuous labels (here $\bar{\mathcal{C}}_i$ is the closure of $\mathcal{C}_i$)

The genealogical structure of excursions

Idea: Glue each excursion component $\mathcal{C}_i$ into a single point.



Formally, for every $a, b \in \mathcal{T}_e$, let

$\widetilde{d}(a,b) =$ total local time at 0 accumulated by Z along the geodesic between a and b,

and set

$$a \approx b \text{ iff } \widetilde{d}(a,b) = 0$$

(holds if a, b belong to the same $\mathcal{C}_i$)

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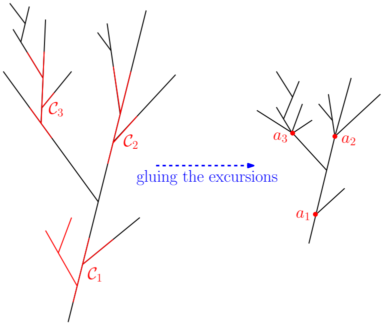

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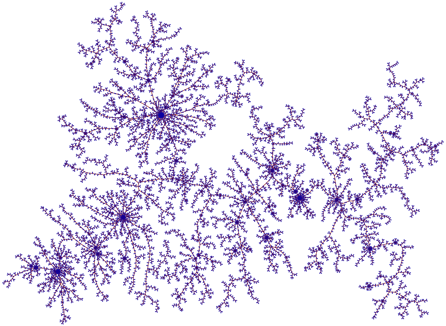
Theorem

$\widetilde{\mathcal{T}} := \mathcal{T}_e / \approx$ equipped with $\widetilde{d}$ is a stable tree with index $3/2$

Each point of infinite multiplicity of $\widetilde{\mathcal{T}}$ is obtained from the gluing of an excursion $\mathcal{C}_i$.

The stable tree with index $3/2$

Constructed as the scaling limit of Galton-Watson trees whose offspring distribution is in the domain of attraction of a stable law with index $3/2$



Simulation by I. Kortchemski

Points have multiplicity
$1, 2$ or ∞

Points of infinite multiplicity are dense

Each point a of infinite multiplicity is assigned a "mass" m_a.
If a is obtained from the gluing of $\mathcal{C}_i$, m_a corresponds to the boundary size of $\mathcal{C}_i$.

The law of excursions

For each "excursion" $\big(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i}\big)$, one can define its boundary size

$$|\partial\mathcal{C}_i| = \lim_{\varepsilon\to 0} \varepsilon^{-2}\,\mathrm{Vol}\big(\{a\in\mathcal{C}_i : |Z_a|<\varepsilon\}\big)$$

Theorem (Abraham-LG)

There exists a σ-finite measure $\mathbb{M}$ (with appropriate scaling properties) on the space of compact $\mathbb{R}$-trees $\mathcal{T}$ equipped with a volume measure $\mathrm{Vol}(\cdot)$ and with labels $(z(a))_{a\in\mathcal{T}}$, such that, conditionally on $(|\partial\mathcal{C}_i|)_{i\in I}$,

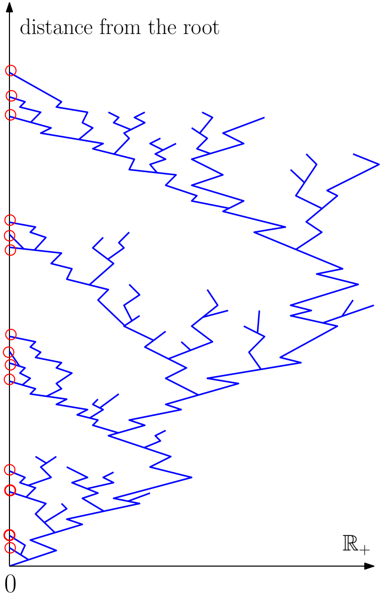
- *the "excursions" $\big(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i}\big)$, $i\in I$ are independent*
- *for every $i\in I$, the distribution of $\big(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i}\big)$ knowing $|\partial\mathcal{C}_i| = r$ is*

$$\mathbb{M}^{(r)} := \mathbb{M}\big(\cdot \mid \Sigma = r\big)$$

where $\Sigma = \lim_{\varepsilon\to 0}\varepsilon^{-2}\,\mathrm{Vol}\big(\{a\in\mathcal{T} : |z(a)|<\varepsilon\}\big)$
(the limit exists $\mathbb{M}$ a.e.)

We can write $\mathbb{M} = \mathbb{M}_+ + \mathbb{M}_-$ and interpret $\mathbb{M}_+$ as a measure on "trees of Brownian paths in $[0,\infty)$". One similarly defines $\mathbb{M}_+^{(r)}$.

The tree of paths under $\mathbb{M}_+$



Under $\mathbb{M}_+$, we now have a tree of nonnegative "Brownian paths" all starting from 0, which stay positive during some interval $(0, \varepsilon]$ and are stopped at the time when they return to 0, if they do return to 0.

Informally, the boundary size Σ counts the number of paths that return to 0 (circled points on the figure).

Explicit formulas under $\mathbb{M}_+$

Joint distribution of boundary size and volume: The distribution of the pair $(\Sigma, \mathrm{Vol}(\mathcal{T}))$ under $\mathbb{M}_+$ has density

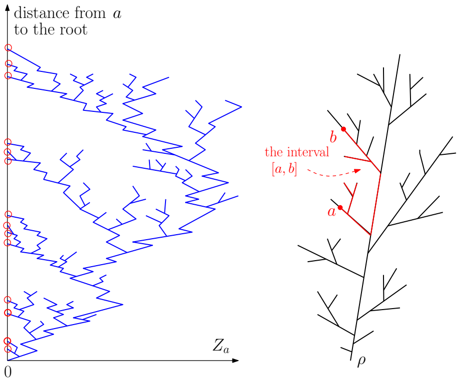
$$f(s,v) = \frac{\sqrt{3}}{2\pi}\,\sqrt{s}\,v^{-5/2}\,\exp\Big(-\frac{s^2}{2v}\Big)$$

As a consequence, for every $s > 0$, the density of $\mathrm{Vol}(\mathcal{T})$ under $\mathbb{M}_+^{(s)} := \mathbb{M}_+(\cdot \mid \Sigma = s\Big)$ is

$$g_s(v) = \frac{1}{\sqrt{2\pi}}\,s^3\,v^{-5/2}\,\exp\Big(-\frac{s^2}{2v}\Big)$$

(this is the asymptotic distribution of the volume of a large random triangulation with a boundary of size n when $n \to \infty$ and the volume is rescaled by n^{-2})

4. The construction of Brownian disks under $\mathbb{M}_+$

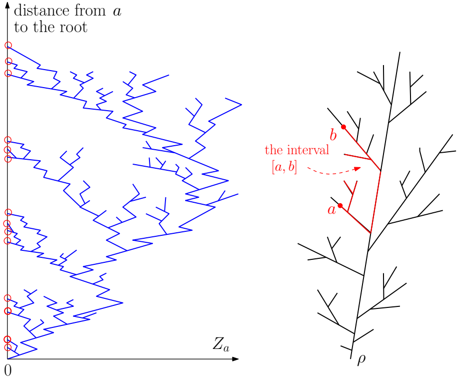


Under $\mathbb{M}_+^{(r)} = \mathbb{M}_+(\cdot \mid \Sigma = r)$,

- we have an $\mathbb{R}$-tree $\mathcal{T}$
- and nonnegative labels $z(a)$, $a \in \mathcal{T}$

Also cyclic order structure on $\mathcal{T}$ that allows one define intervals $[a, b]$ (informally, points visited when going from a to b around the tree).

4. The construction of Brownian disks under $\mathbb{M}_+$



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For $a, b \in \mathcal{T}$, set

$$D^\circ(a,b) = z(a) + z(b) - 2\max\Big\{\min_{c\in[a,b]} z(c), \min_{c\in[b,a]} z(c)\Big\}.$$

Imitating the construction of the Brownian sphere would require identifying a and b if $D^\circ(a,b) = 0$. But here this would mean identifying all boundary points (all c such that $z(c) = 0$)!

Constructing free Brownian disks

Recall $D^\circ(a,b) = z(a) + z(b) - 2\max\Big\{\min_{c\in[a,b]} z(c), \min_{c\in[b,a]} z(c)\Big\}$.

Set $\partial\mathcal{T} = \{c \in \mathcal{T} : z(c) = 0\}$, $\mathcal{T}^\circ = \mathcal{T}\backslash\partial\mathcal{T}$ and, for $a, b \in \mathcal{T}^\circ$,

$$\Delta^\circ(a,b) = \begin{cases} D^\circ(a,b) & \text{if } \max\Big\{\min_{[a,b]} z(c), \min_{[b,a]} z(c)\Big\} > 0, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\Delta(a,b) = \inf_{\substack{a=a_0,a_1,\ldots,a_k=b \\ a_i\in\mathcal{T}^\circ}} \sum_{i=1}^{k} \Delta^\circ(a_{i-1}, a_i).$$

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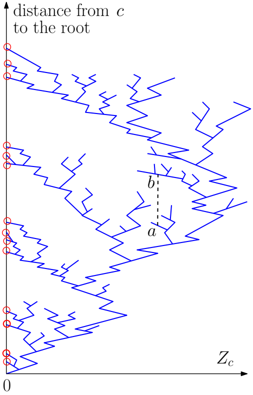
Theorem

Under $\mathbb{M}_+^{(r)}$, $(\Delta(a,b), a, b \in \mathcal{T}^\circ)$ has a continuous extension to $\mathcal{T}\times\mathcal{T}$, which is a pseudo-metric on $\mathcal{T}$. The associated quotient space $\mathbb{D}$ equipped with the distance induced by Δ is a free Brownian disk with perimeter r.

Remark: $\partial\mathbb{D}$ corresponds to $\partial\mathcal{T}$ in the quotient space.

Uniform measure on the boundary



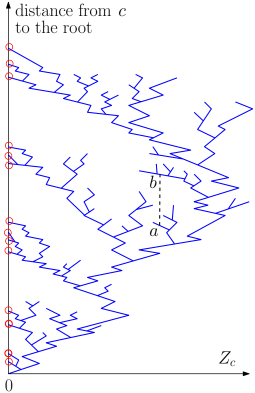
Interpretation: We glue $a, b \in \mathcal{T}^\circ$ if

- they have the same label $z(a) = z(b) > 0$
- going from a to b "around" the tree $\mathcal{T}$ one encounters only vertices with greater label.

The Bettinelli-Miermont construction also relied on using a labeled forest, but here we have the additional remarkable interpretation of labels:

$z(c) = \Delta(c, \partial\mathbb{D})$ coincides with the distance from (the equivalence class of) c to $\partial\mathbb{D}$.

Uniform measure on the boundary



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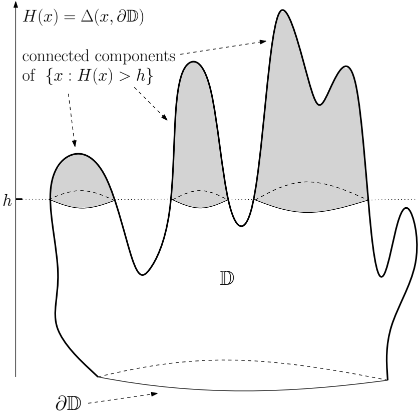
One can use this to construct the uniform measure on the boundary.

Proposition

The formula $\langle \mu, \varphi \rangle = \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, \varphi(x)\, \mathbf{1}_{\{\Delta(x,\partial\mathbb{D})<\varepsilon\}}$
defines a finite measure on the boundary with total mass r.

5. Cutting Brownian disks at heights



$(\mathbb{D}, \Delta)$ is the free Brownian disk with perimeter r

For $x \in \mathbb{D}$, $H(x) = \Delta(x, \partial\mathbb{D})$ is called the height of x.

Fix $h > 0$. For each connected component $\mathcal{C}$ of $\{x : H(x) > h\}$, can define its boundary size (perimeter)

$$|\partial\mathcal{C}| = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \mathrm{Vol}(\{x \in \mathcal{C} : H(x) < h + \varepsilon\})$$

Theorem (LG-Riera)

Conditionally on their boundary sizes, the connected components of $\{x \in \mathbb{D} : H(x) > h\}$, equipped with their intrinsic metrics, are independent free Brownian disks with the prescribed perimeters.

Question. How does the collection of perimeters of connected components of $\{x \in \mathbb{D} : H(x) > h\}$ evolve as h varies ?

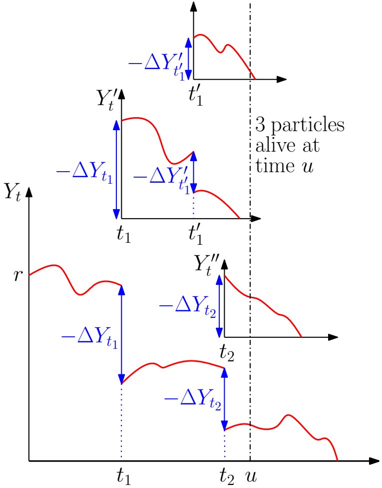
Write $\mathcal{C}^{1,h}, \mathcal{C}^{2,h}, \ldots$ for the connected components of $\{x \in \mathbb{D} : H(x) > h\}$ ranked in decreasing order of their boundary sizes, and

$$\mathbf{X}(h) = (|\partial\mathcal{C}^{1,h}|, |\partial\mathcal{C}^{2,h}|, \ldots)$$

The preceding theorem suggests that $(\mathbf{X}(h))_{h\geq 0}$ satisfies a kind of branching property analogous to that of growth-fragmentation processes.

Growth-fragmentation processes

Basic ingredient: Y self-similar Markov process with values in $\mathbb{R}_+$ and only negative jumps, absorbed at 0.



The process starts with an initial particle (Eve particle) whose mass evolves in time according to the law of Y started at r.
When the mass of the initial particle has a (negative) jump of size $-\delta$, a new particle (child of the Eve particle) is created, whose mass then evolves according to the law of Y started at δ.
In turn, each child of the Eve particle has children at jump times of its mass process, and so on.

The associated growth-fragmentation process is:

$\mathbf{Y}(t) =$ ranked sequence of masses of particles alive at time t.

Growth-fragmentation process in the Brownian disk

Recall that $\mathbb{D}$ is the free Brownian disk with perimeter r, and

$$\mathbf{X}(h) = (|\partial\mathcal{C}^{1,h}|, |\partial\mathcal{C}^{2,h}|, \ldots)$$

Here $\mathcal{C}^{1,h}, \mathcal{C}^{2,h}, \ldots$ are the connected components of $\{x \in \mathbb{D} : H(x) > h\}$.

Theorem (LG-Riera)

$(\mathbf{X}(h))_{h\geq 0}$ *is a growth-fragmentation process whose Eve particle mass process* X *(starting from* 1*) can be obtained as follows:*

$$X_t = \exp(\xi_{\tau(t)}),$$

where

$$\tau(t) = \inf\left\{u \geq 0 : \int_0^u e^{\xi_s/2}\,\mathrm{d}s > t\right\}$$

and ξ *is the spectrally negative Lévy process with Laplace exponent*

$$\psi(q) = \sqrt{\frac{3}{2\pi}}\left(-\frac{8}{3}q + \int_{1/2}^{1} (x^q - 1 + q(1-x))\,(x(1-x))^{-5/2}\,\mathrm{d}x\right).$$

Remarks

- The formula

$$X_t = \exp(\xi_{\tau(t)})$$

is the Lamperti representation of a self-similar Markov process in terms of a Lévy process.

- The theorem is closely related to the work of Bertoin, Curien, Kortchemski who studied asymptotics for a discrete analog of the process $\mathbf{X}(h)$ (for triangulations with a boundary).

- The measure

$$(x(1-x))^{-5/2}\,\mathrm{d}x$$

that appears in the formula for ψ should be compared with the dislocation measure $(x(1-x))^{-3/2}\,\mathrm{d}x$ corresponding to the (pure) fragmentation process obtained by cutting the Brownian tree at heights (Bertoin).